

Various perspectives on the design and assessment of structures using the redundancy matrix and its homogenization

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Abstract

The redundancy matrix quantifies the distribution of static indeterminacy in a structure and is therefore a measure of the internal constraint. It thus describes the importance of the individual elements for the load-bearing behavior. The trace of the matrix is equal to its rank and it corresponds to the degree of static indeterminacy of the entire structure. Ströbel stated in his doctoral thesis that “the redundancy should be distributed equally across all elements” to achieve a robust structure that has load-bearing reserves in the event of element failure. In this contribution, this statement will be examined from various perspectives in more detail by describing the mathematical and mechanical properties of the redundancy matrix and how changes in the redundancy distribution affect the load-bearing behavior of the structure. Aspects such as robustness, performance capabilities, and constraint load cases will be discussed. In addition, it is investigated how a homogenized redundancy distribution can be obtained during a design or optimization procedure. The homogenization of the redundancy matrix using size and shape optimization already showed improvements in terms of robustness. This is investigated in more detail by introducing additional constraints, such as a prescribed total mass of the structure, opening up the possibilities to further improve structural performance. In the end, an outlook on large spatial structures and further challenges is given.

Keywords: redundancy matrix, structural performance, design, optimization, statical indeterminacy

1. Introduction

To evaluate the performance of load-bearing structures in buildings, well-known indicators are displacements, stress and strain. In building codes all of them have to be lower than a certain threshold to have enough performance capabilities, resistance and to be on the safe side for the ultimate limit state and for the serviceability limit state. In contrast to this, knowledge about static indeterminacy and robustness is barely quantified and not required by the building codes. Robustness as a performance indicator is stated by Ströbel in his doctoral thesis [1] and can be quantified by the distribution of static indeterminacy, which is equal to the trace of the redundancy matrix.

A detailed derivation of the redundancy matrix can be found in [2], and the first use cases of a homogeneously distributed redundancy matrix to optimize structural properties, such as robustness and assemblability, are made by Forster and von Scheven [3]. While they consider an individual load case and only one single imperfection length, these properties could potentially be further improved by prescribing the structural mass or additional structural constraints [3]. Other robustness approaches are proposed by Starossek and Haberland [4] and by Kou et al. [5], which result in scalar quantities related to the redundancy matrix. Geiger [6] describes a use-case of self-stress states for adaptive structures.

The redundancy matrix and related matrices in structural mechanics, as well as the properties and interpretations of the redundancy matrix are described based on a truss example in Section 2. Section 3

describes two optimization methods to homogenize the redundancy distribution and compare the results of both with the original structure. Afterwards, a large-scale structure is investigated before the paper is summarized in Section 5, and an outlook for further investigations is given in Section 6.

2. Definition, properties and interpretations of the redundancy matrix

2.1. Matrix structural analysis for discrete structures

Structural analysis for discrete structures can be written in matrix-vector format. The following introduces the relevant fundamentals and notation throughout the contribution. The number of load-carrying modes per element n_m is equal to one for truss elements and relates to the axial elongation. It is equal to three for plane beam elements, which also includes shear and bending. If the structure consists of only one type of element, the total number of load-carrying modes is calculated by $n_l = n_m n_e$, where n_e denotes the number of elements. The redundancy matrix $\mathbf{R} \in \mathbb{R}^{n_l \times n_l}$ is a projection matrix which maps pre-deformations $\mathbf{e}_0 \in \mathbb{R}^{n_l}$ on elastic deformations

$$\mathbf{e}_{el} = -\mathbf{R}\mathbf{e}_0. \quad (1)$$

The elastic deformations $\mathbf{e}_{el} \in \mathbb{R}^{n_l}$ are related to stress resultants $\mathbf{s} \in \mathbb{R}^{n_l}$ by the diagonal member stiffness matrix $\mathbf{C} \in \mathbb{R}^{n_l \times n_l}$,

$$\mathbf{s} = \mathbf{C}\mathbf{e}_{el}. \quad (2)$$

The redundancy matrix is defined by von Scheven et al. [2] as follows

$$\mathbf{R} = \mathbf{I} - \mathbf{A}(\mathbf{A}^\top \mathbf{C} \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{C}, \quad (3)$$

where $\mathbf{A} \in \mathbb{R}^{n_l \times n_d}$ is the compatibility matrix, where n_d denotes the number of degrees of freedom. Its transpose $\mathbf{A}^\top \in \mathbb{R}^{n_d \times n_l}$ is the equilibrium matrix and describes the relationship between external forces and internal stress resultants. The term in parentheses of eq. (3) can be identified as the stiffness matrix

$$\mathbf{K} = \mathbf{A}^\top \mathbf{C} \mathbf{A}. \quad (4)$$

2.2. Mathematical properties and mechanical interpretations of the redundancy matrix

The main diagonal entries in the redundancy matrix, R_{ii} , are equal to the redundancy r_i of that element and the sum of them is equal to the static indeterminacy of the structure n_s

$$\sum_{i=1}^{n_e} R_{ii} = \sum_{i=1}^{n_e} r_i = n_s. \quad (5)$$

Thus, the degree of static indeterminacy n_s is no longer an integer value, but a distributed quantity, and the redundancy matrix shows how the static indeterminacy n_s is distributed in the structure.

From a mathematical point of view, the redundancy matrix $\mathbf{R}$ is a projection matrix. Projection matrices are idempotent, which means that $\mathbf{R}\mathbf{R} = \mathbf{R}$, the eigenvalues are $\lambda_i \in \{0, 1\}$ and their rank $\text{rk}(\mathbf{R})$ is equal to their trace $\text{tr}(\mathbf{R})$

$$\text{rk}(\mathbf{R}) = \text{tr}(\mathbf{R}) = n_s. \quad (6)$$

Projection matrices are diagonalizable matrices and the multiplicity of $\lambda = 1$, which is also equal to n_s , indicates the number of independent self-stress states. Their eigenvectors span the space of self-stress, described in [2].

Usually, $\mathbf{R}$ is an *oblique* projection matrix. It results in an *orthogonal* projection matrix if $\mathbf{R} = \mathbf{R}^\top$, which only occurs if the member stiffness matrix is a multiple of the identity matrix, $\mathbf{C} = c\mathbf{I}$.

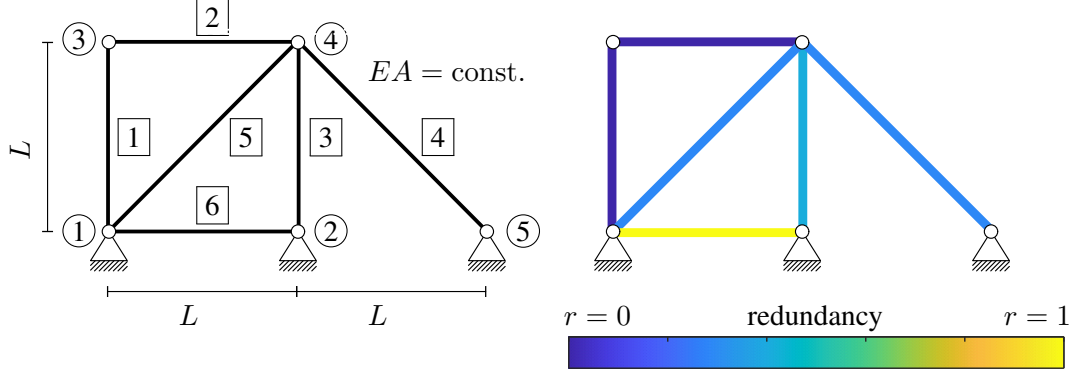


Figure 1: Truss structure with 6 elements and 5 nodes (left) and colored redundancy distribution (right)

Following eq. (3), if the stiffnesses of the elements change, the redundancy distribution changes and thus can be homogenized by a targeted variation of $\mathbf{C}$ and $\mathbf{A}$. Each column of the redundancy matrix generally refers to one load-carrying mode, thus, in truss structures, to one element. For the truss structure in Fig. 1 the member stiffness matrix $\mathbf{C}$ is given in eq. (7) and the corresponding redundancy matrix $\mathbf{R}$ in eq. (8)

$$\mathbf{C} = \text{diag} \left(\frac{E_i A_i}{L_i} \right) = \text{diag} \left(\frac{EA}{L} \quad \frac{EA}{L} \quad \frac{EA}{L} \quad \frac{EA}{\sqrt{2}L} \quad \frac{EA}{\sqrt{2}L} \quad \frac{EA}{L} \right), \quad (7)$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{0.000} & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & \mathbf{0.000} & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & \mathbf{0.414} & -0.293 & -0.293 & 0.000 \\ 0.000 & 0.000 & -0.414 & \mathbf{0.293} & 0.293 & 0.000 \\ 0.000 & 0.000 & -0.414 & 0.293 & \mathbf{0.293} & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & \mathbf{1.000} \end{bmatrix}, \quad n_s = \text{tr}(\mathbf{R}) = 2. \quad (8)$$

As the first two elements are a statically determinate substructure and the sixth element is fully redundant, changing the stiffness of these elements has no effect on the redundancy values R_{ii} .

2.3. Self-stress states

To obtain a basis for the space of self-stress states, the subspaces can be used

$$\text{null}(\mathbf{A}^\top) = \text{im}(\mathbf{R}^\top). \quad (9)$$

The first term in eq. (9) describes the null space of the equilibrium matrix [7] and the second term refers to the image of the transposed redundancy matrix [6]. The eigenvectors Φ_i for the self-stress states corresponding to the eigenvalues equal to 1 ($\lambda_i = 1$) for the introductory example in Fig. 1 are

$$\Phi_1 = [0 \quad 0 \quad -0.7071 \quad 0.5 \quad 0.5 \quad 0]^\top, \quad \Phi_2 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^\top, \quad (10)$$

and they are visualized in Fig. 2. Applying these eigenvectors as initial deformations $\mathbf{e}_0$ does not lead to further deformations, but to pure stress states in equilibrium. The second vector in eq. (10) only influences the sixth element, which is fixed between the boundary conditions and thus is fully redundant. It does not contribute to the load-bearing behavior and does not influence the other elements. Φ_1 influences elements 3, 4 and 5. Elements 1 and 2 are statically determinate, therefore, internal stress results only through external forces.

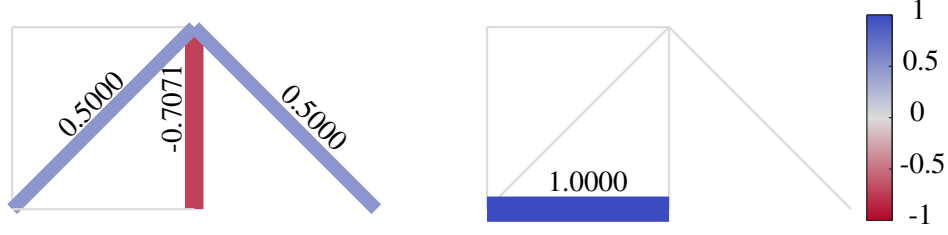


Figure 2: Self-stress states colored for the introductory example, thickness scaled by values (see eq. (10))

2.4. Constraint load cases

Constraint load cases like temperature loads, geometric imperfections, and non-zero prescribed displacements at the supports induce self-stress in statically indeterminate substructures. These states of self-stress can be described with the help of the redundancy matrix; see eq. (9). The results of constraint load cases are discussed in the following. First, a temperature load ΔT_1 is applied to element 3. The resulting prescribed elongation $e_{0,3}$ leads to elastic elongations e_{el} in elements 3, 4 and 5. These elastic elongations can be read from the third column of the redundancy matrix eq. (8) and the resulting normal forces can be calculated using eq. (2). The statically determinate elements 1 and 2 and element 6 are not affected by this load case. For comparison, the same load magnitude ΔT_1 is applied on element 4. Since element 4 is $\sqrt{2}$ times longer than element 3, the prescribed elongation is $\sqrt{2}$ times greater than in the first load case. Since element 4 is softer by the same factor the resulting normal forces are the same as for the first load case. When considering a temperature load $\Delta T_2 = \frac{\Delta T_1}{\sqrt{2}}$, the prescribed elongation would be the same as for the first load case and the resulting normal forces would be smaller. These investigations are similar when looking at manufacturing imperfections. If element 4 is manufactured with an error of 1 cm the self-stress is smaller compared to the case when element 3 is manufactured with an error of 1 cm. Constraint load cases on the statically determinate elements 1 and 2 yield only deformation of the corresponding element but do not induce any stress into the structure. A constraint load on element 6 results in forces only and no deformations as both end nodes are fixed, see Fig. 2. The redundancy entries themselves are a measure for the imperfection sensitivity [3].

As the trace of the redundancy matrix during change in size or shape remains constant, only a redistribution of values R_{ii} is possible. A homogenized distribution means that every element has the same amount of constraint from the surrounding structure. This can be used to decrease or remove peaks in imperfection induced strain and stress. To determine the displacements $\mathbf{d} \in \mathbb{R}^{n_d}$ in terms of the degrees of freedom for given pre-deformations $\mathbf{e}_0$ in the elements eq. (11) can be used

$$\mathbf{d} = \mathbf{K}^{-1} \mathbf{A}^\top \mathbf{C} \mathbf{e}_0. \quad (11)$$

For the introductory example, this leads to the following displacements

$$\mathbf{d} = \begin{bmatrix} 0 & -1 & 0 & -0.707 & 0.707 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.707 & 0.707 & 0 \\ 0 & 0 & 0.586 & 0.293 & 0.293 & 0 \end{bmatrix} \mathbf{e}_0. \quad (12)$$

Fig. 3 shows the deformed structure for exemplary pre-deformations, resulting from geometric imperfections or temperature loads. In the first case, since element 1 is statically determinate, it is not subject to any constraint and can freely elongate, see Fig. 3 left. The same applies to element 2. In contrast, element 6 is completely fixed between two supports, which is why any initial deformation is fully reversed. For element 3, expansion is possible and results in a deformation $e_3 = e_{0,3}(1 - r_3)$, Fig. 3 in the middle. Analogously, the same applies to element 5 (Fig. 3 right) and to element 4, which are also mirror images of each other. Element 2 in the third case undergoes only a rigid-body rotation,

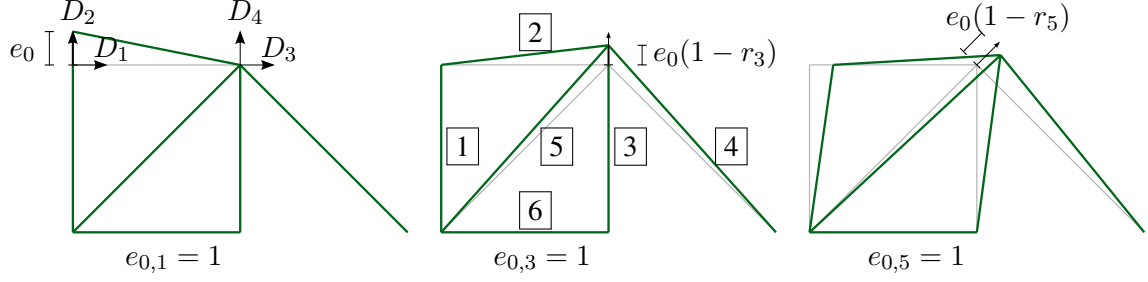


Figure 3: Three exemplary displaced structure for pre-deformation e_0 according to eq. (12)

while elements 4 and 5 deform according to the off-diagonal terms in the corresponding columns of $\mathbf{R}$, which generate deformations $e_{0,i}(1 - R_{ij})$.

2.5. Robustness

There is a strict link between a common robustness measure [4] and the redundancy values

$$R_{ii} = \frac{\det(\bar{\mathbf{K}}_i)}{\det(\mathbf{K})}, \quad (13)$$

as derived in [3]. The redundancy value R_{ii} is equal to the ratio between the determinant of the stiffness matrix after the failure of element i , $\bar{\mathbf{K}}_i$, and the determinant of the original stiffness matrix $\mathbf{K}$. This robustness metric in eq. (13) indicates robustness according to [1] because a structure is optimally robust when the redundancy values are homogeneously distributed. This means that every element is equally important and is backed up by the same amount of structural “reserves” if it fails [1]. Otherwise, for the limit case $R_{ii} = 1$ the damage of the element has no consequence, and for $R_{ii} = 0$ the failure leads to a singular stiffness matrix, what at least indicates a kinematic substructure.

3. Homogenization approach of the redundancy matrix

Several optimization methods can achieve homogenization of the redundancy distribution. In this contribution, two approaches are examined: Size optimization is carried out by updating the member stiffness matrix $\mathbf{C}$, while shape optimization changes the geometry by moving nodes, resulting in changes in the compatibility matrix $\mathbf{A}$ as well as in the member stiffness matrix $\mathbf{C}$.

3.1. Size optimization

The size optimization is done with an iterative algorithm based on a students thesis eq. (14), which scales the member stiffness matrix $C_{ii}^{(n+1)}$ in iteration $(n + 1)$ by the ratio of the corresponding redundancy value $R_{ii}^{(n)}$ in the current iteration n with respect to the target redundancy R_{aim} , they aim for.

$$C_{ii}^{(n+1)} = \frac{R_{ii}^{(n)}}{R_{\text{aim}}} C_{ii}^{(n)}, \quad \text{where } R_{\text{aim}} = \frac{n_s}{n_l} = \frac{n_s}{n_e n_m}, \quad (14)$$

$$\mathbf{C}^{(n+1)} = \text{diag} \left(C_{ii}^{(n+1)} \right).$$

Without the target redundancy in the denominator, the entries in $\mathbf{C}$ would decrease in every iteration step n , eventually leading to numerical zeros in $\mathbf{C}$ for large n . For small n this is just a decrease in stiffness, which could be compared to a reduction in mass or stiffness or rescaled at the end to reasonable values. An element redundancy greater than the target redundancy $R_{ii} > R_{\text{aim}}$ means an increase in stiffness that makes the element more important and less redundant, and $R_{ii} < R_{\text{aim}}$ vice versa. The special case $R_{ii} = 0$ would be equivalent to element removal. However, since a statically determinate substructure has no influence on the redundancy distribution, updating such elements can be ignored, similarly to completely redundant ones.

The update algorithm in eq. (14) can be interpreted for truss structures as a change in stiffness by either changing the Young's modulus E or the area of the cross section A . For beams, the change in the stiffnesses also influences the cross-sectional shape by varying the values of the moment of inertia I_y and I_z as well as the torsional inertia I_T . Structures with $n_s = 1$ converge after one iteration to the final solution. With higher degree of static indeterminacy n_s and more modes n_m , the number of iterations increases and the absolute gain decreases in every step. For the example in Fig. 1 the homogenized redundancy distribution ends up as

$$\mathbf{R}^h = \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0.333} & -0.236 & -0.236 & 0 \\ 0 & 0 & -0.471 & \mathbf{0.333} & 0.333 & 0 \\ 0 & 0 & -0.471 & 0.333 & \mathbf{0.333} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}. \quad (15)$$

The member stiffness matrix entries change according to the algorithm to $C_{33} = 2C_{44} = 2C_{55}$.

3.2. Shape optimization

Optimizing the redundancy distribution via relocating nodal positions can be done manually or with optimization algorithms, for instance gradient based ones like *Sequential quadratic programming* (SQP), where the design variables are the nodal coordinates. The objective function of the optimization problem is formulated in general as follows

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad f(\mathbf{x}) = \Delta R = R_{\max} - R_{\min}. \quad (16)$$

The introductory example in Fig. 1 with the vertical coordinate of node 2 as design variable $x = y_2$ needs additional constraints, because the objective function is non-decreasing due to the two fixed boundaries $R_{\min} = 0$, $R_{\max} = 1$. Therefore, these special cases should be excluded, which results in the constraints for the calculation of the objective function $R_{\min} > 0$, $R_{\max} < 1$. Moving node 2 in vertical direction by $L(1 - 1/\sqrt{2})$ to shorten element 3 results in the same homogenized redundancy matrix as given in eq. (15), because the element stiffness is $C_{33} = 2C_{44} = 2C_{55}$ and $\mathbf{A}$ does not change here. Shape optimization according to eq. (16) is studied in [3] to increase robustness in addition to improving assemblability of the structure, while decreasing imperfection induced strains during assembly. Further enhancement is possible by including additional constraints, such as that the mass remains constant or that some eigenvalues of the stiffness matrix must not decrease, which is not considered in [3], although the structures after optimization show improved performance compared to the original structure. For the simple case of the introductory example, the constraint on the eigenvalues of the stiffness matrix is fulfilled even without imposing it in the optimization process.

3.3. Comparison of results

3.3.1. Performance capabilities

For performance comparison, the eigenvalues of the stiffness matrix will be studied, which are a load-independent indicator for structures, and the stiffness matrix is part of the redundancy, see eq. (3). The eigenvalues of the stiffness matrix $\mathbf{K}$ for the introductory example are

$$\lambda_{\mathbf{K}}^0 = EA \cdot \left[1 - \frac{\sqrt{2}}{2} \quad 1 \quad 1 + \frac{\sqrt{2}}{2} \quad 1 + \sqrt{2} \right]^T \approx EA \cdot [0.2929 \quad 1 \quad 1.707 \quad 2.414]^T, \quad (17)$$

while for the shape-optimized structure, the eigenvalues are

$$\lambda_{\mathbf{K}}^h = EA \cdot \left[1 - \frac{\sqrt{2}}{2} \quad 1 \quad \sqrt{2} + \frac{\sqrt{2}}{2} \quad 1 + \sqrt{2} \right]^T \approx EA \cdot [0.2929 \quad 1 \quad 2.121 \quad 2.414]^T. \quad (18)$$

The third eigenvalue increases by approximately 24%, which means a higher stiffness of the corresponding deformation mode. Therefore, more force is needed to produce displacements in the direction of this eigenvector. The other eigenvalues remain the same, which means that there is no overall decrease in stiffness. The determinant is $\det \mathbf{K} = \frac{1+\sqrt{2}}{2}(EA)^4 \approx 1.2071(EA)^4$, while for the shape-optimized structure it increases to $\det \mathbf{K}^h = 1.5(EA)^4$. In addition, the mass is reduced by approximately 4%. For the size-optimized case, it depends on if and how the mass will be redistributed. By rescaling to the original mass, one could scale the vanished mass equally among all elements or only scale the elements with certain redundancies, like keeping the statically determinate and fully redundant elements at the starting value. The cross-sectional areas are already optimized after one iteration of the algorithm, as the degree of static indeterminacy without the fully redundant element is only 1, $A^{\text{opt}} = A \cdot [1 \ 1 \ 1.2426 \ 0.8787 \ 0.8787 \ 1]$, where A is the cross-sectional area of the original structure. This results in the following eigenvalues of the stiffness matrix

$$\lambda_{\mathbf{K}}^{\text{opt}} = EA \cdot [0.2635 \ 1 \ 1.864 \ 2.358]^T. \quad (19)$$

Compared to the eigenvalues of the original structure eq. (17), the first eigenvalue decreases by 11%, the last by 3%, while the third increases by 9%. The total mass decreases by approximately 2%. The determinant decreases slightly to $\det \mathbf{K}^{\text{opt}} = 1.1582(EA)^4$.

3.3.2. Self-stress states

For the size-optimized structure, there is no change in the equilibrium matrix $\mathbf{A}^T$, therefore, the subspace null $(\mathbf{A}^T)$ is equal to the original structure and the eigenvectors Φ_i of the self-stress states look the same, see eq. (10) and Fig. 2, respectively. For the shape-optimized structure, the self-stress states are visualized in Fig. 4. As the nodal coordinates change at the fixed support, therefore, at fixed degrees of freedom, which is usually not the case, both the equilibrium matrix and the values of the self-stress states stay the same. Homogenization in this case does not have an influence here. In general, during shape-optimization process, when $\mathbf{A}^T$ and $\mathbf{R}^T$ change, the subspaces, and thus the self-stress states also change. It is unclear whether homogenization of $\mathbf{R}$ also leads to a more uniform distribution of the states of self-stress. However, if that is the goal, optimization can also be performed on the relevant subspaces.

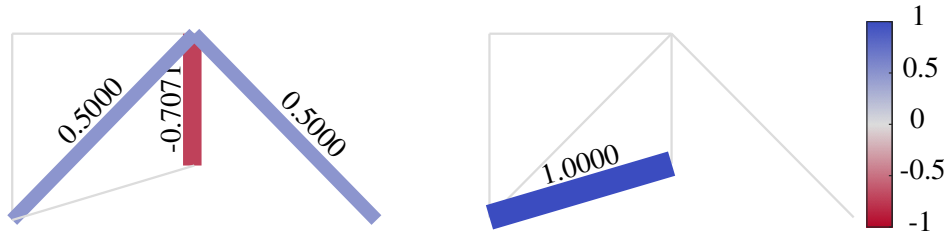


Figure 4: Self-stress states colored for the shape-optimized structure in the introductory example, thickness scaled by values

3.3.3. Robustness

The robustness of the structure can only be optimized for elements 3, 4, and 5 by redistributing the redundancy. Maximizing the redundancy of these elements minimizes the effect of failures on the load-bearing behavior, but as the sum of the redundancy is limited, homogenization of the redundancy distribution can be seen as the maximized redundancy independent of the failed element and therefore a robust design, as stated in [3]. It results in equal importance and reserves for all elements [1]. For the introductory example, this means a lower redundancy of element 3 but an increased redundancy of elements 4 and 5.

3.3.4. Constraint load cases

For the shape-optimized structure shown in Fig. 4, the displacements due to constraint loads can be calculated as

$$\mathbf{d}^h = \left(\mathbf{K}^h\right)^{-1} \mathbf{A}^\top \mathbf{C}^h \mathbf{e}_0 = \begin{bmatrix} 0 & -1 & 0 & -0.707 & 0.707 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.707 & 0.707 & 0 \\ 0 & 0 & 0.667 & 0.236 & 0.236 & 0 \end{bmatrix} \mathbf{e}_0. \quad (20)$$

In comparison to the displacements of the reference structure in eq. (12), the changes arise only from the change in the member stiffness matrix $\mathbf{C}^h$ and the equilibrium matrix $\mathbf{A}^\top$ remains the same. The only changes are in the fourth row. The fourth degree of freedom is in the same direction as the elongation mode of element 3. As the value increases ($0.667 > 0.586$), the displacement after constraint load cases on element 3 is larger than for the reference structure. This can be explained with the decreased redundancy ($0.333 < 0.414$, see eq. (8)). As the redundancies of elements 4 and 5 increase through shape optimization, the last entry of columns 4 and 5 in eq. (20) decrease, because the elements are more constraint. Elements 1 and 2 can expand freely and element 6 is fully constrained. For the strain and stress level of the elements, eq. (1) and eq. (2) can be used, where these distributions can be seen column-wise [3].

4. Large-scale spatial structure

The homogenization of the redundancy distribution according to Section 3 will be studied next for the large-scale spatial truss structure depicted in Fig. 5, which was also investigated in [2].

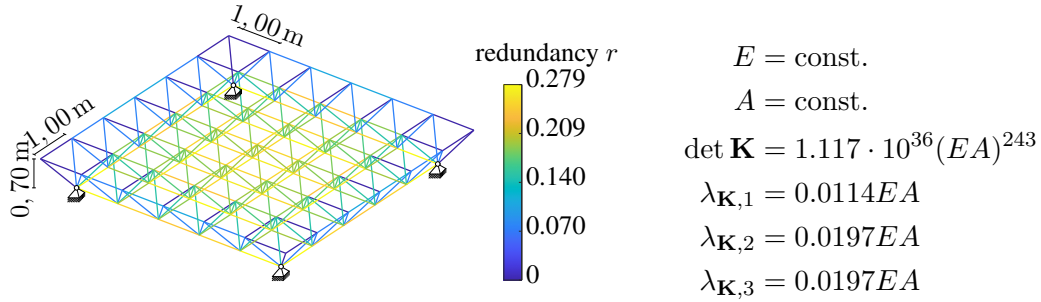


Figure 5: Mero roof with colored redundancy distribution

The degree of static indeterminacy is $n_s = 45$ and the redundancy r_i varies between 0 and 0.279, the number of nodes is 85, the number of elements is equal to 288, including 32 elements with $r_i = 0$.

4.1. Size optimization

The target redundancy for the size optimization algorithm is $R_{\text{aim}} = 45/288 = 0.156$. A completely homogenized redundancy distribution by size optimization is not possible as the elongation mode between two supports is fixed. For these five elements of each lower outer edge the redundancy $r_i = 1/5 = 0.2$. This would be the final minimal ΔR_{ii} as some elements have zero redundancy, see Fig. 6. The resulting difference between the redundancy values is $\Delta R_{ii} = 0.206$ after 11 iterations and thus quite close to the theoretical solution of $\Delta R_{ii} = 0.2$. To further converge to this theoretical solution, the nodal positions have to move out of line to form an arc. It can be seen in Fig. 6 on the right that the curve for ΔR_{ii} flattens out with an increasing number of iterations n and in every step the gain decreases as already mentioned in Section 3.1. Furthermore, the determinant of the stiffness matrix decreases rapidly and may end up with a numerical zero, as more iterations result in domains with small values for the cross-sectional areas A_i and small eigenvalues $\lambda_{\mathbf{K},i}$, even after rescaling to the starting mass, which is an undesirable behavior. The first three eigenvalues of the stiffness matrix in

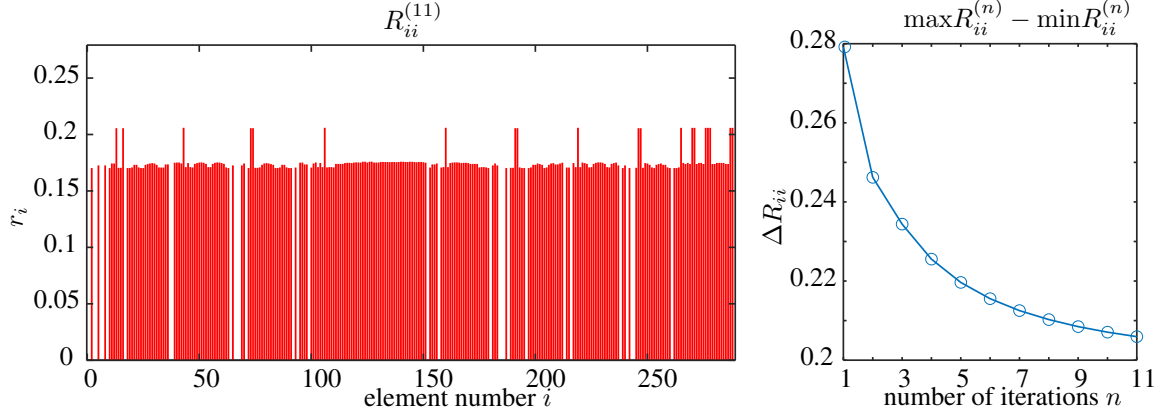


Figure 6: Size-optimized redundancy values r_i of every element i in iteration $n = 11$ and ΔR_{ii} in every iteration n of the algorithm for the mero roof

the 11th iteration $\lambda_{\mathbf{K}}^{\text{h},\text{rsc}} = EA \cdot [0.0033 \ 0.0052 \ 0.0052]$ are lower than the first eigenvalue $\lambda_{\mathbf{K},1}$ of the original structure. This problem results from the statically determinate substructures and the support conditions, as the target redundancy cannot be reached by all elements, which leads to an imbalance the algorithm cannot resolve by stiffness redistribution. The cross-sections can be bounded with some limits for $A_{\min}$ and $A_{\max}$ to treat the too flexible behavior, but this influences the performance of the algorithm. However, this trade-off has to be treated by the designer.

4.2. Shape optimization

The structure shown in Fig. 7 results from shape-optimization using symmetry and therefore only one quarter of the domain. The support position is fixed, and the nodes at the top along the symmetry axes remain on the axes, resulting in 64 spatial coordinates as design variables $\mathbf{x}^{64 \times 1}$. An *SQP* algorithm is used, and the mass is kept constant during the optimization procedure as an equality constraint. The optimization converges after 164 iteration steps to a minimum of $\Delta R = 0.2157$ with $R_{\min} = 0$, which is a bit larger compared to the value obtained by size optimization. The first three eigenvalues $\lambda_{\mathbf{K}}^{\text{h}}$ are similar, and the determinant of the stiffness matrix increases by roughly one order of magnitude, thus, increasing in this case larger eigenvalues for better performance. Robustness increases as the distribution is more even, but the statically determinate substructures cannot be eliminated, neither by size nor shape optimization.

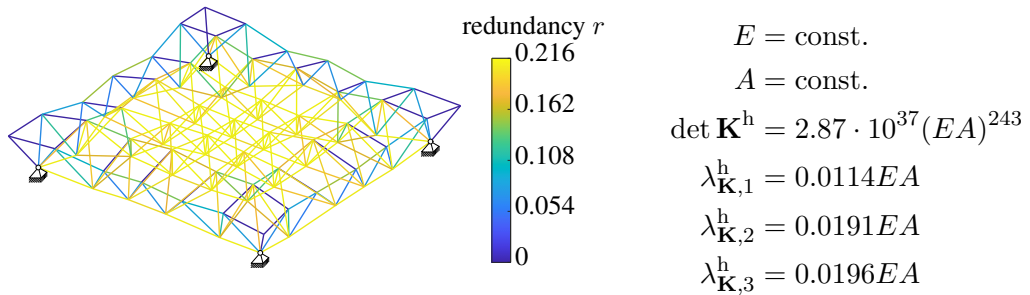


Figure 7: Mero roof with colored redundancy distribution after shape optimization

5. Conclusion

In conclusion, the redundancy matrix and the distribution of static indeterminacy are suitable tools for the design process to optimize designs, particularly at an early stage. Measurable differences are also possible with regard to constraint load cases, and designs can be optimized accordingly. For

self-stress states, specific subspaces must be optimized or can highlight non-critical elements. The size optimization algorithm is straightforward but still has some issues, particularly, if an imbalance of the mechanically possible redundancy and the target redundancy exists, and in large systems with high static indeterminacy, where many steps are needed, which can lead to local instabilities. There is still potential for improvement here. However, as an initial approach or with a limited number of steps or restrictions on cross-sectional variation, these problems can be reduced or even eliminated. Shape optimization is more complex to implement, as the choice and number of design variables are crucial, but it leads to better performing structures compared to size optimization.

6. Outlook and arising challenges

When using beam elements, the number of load-bearing mechanisms per element increases, which also increases the size of the matrices and the numerical effort. Furthermore, it is necessary to consider how homogenization should be carried out in this context: Should it be performed at the element level or at the mode level? These considerations are useful in relation to the physical values of the cross-sectional parameters. It is also possible to homogenize modes with different target redundancies, thus aiming for lower redundancies, e.g. in the elongation mode compared to the bending or torsional mode.

While this contribution has examined optimization in terms of either size or shape, it is also possible to combine both optimization strategies. It is also conceivable to carry out a topology optimization, in particular to eliminate statically determinate substructures or to remove completely redundant elements. Since the eigenvalues λ_i of the redundancy matrix are only zero and one, it might be interesting to examine the singular values σ_i as they are not necessarily equal to one or zero and exhibit an inherently decreasing order that is not exhibited by the eigenvalues. The corresponding singular vectors might give additional information. This is currently being investigated and has already been done in [7] for the equilibrium matrix.

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References

- [1] D. Ströbel, *Die Anwendung der Ausgleichsrechnung auf elastomechanische Systeme*, dissertation, Universität Stuttgart, Germany, Deutsche Geodätische Kommission bei der Bayerischen Akademie der Wissenschaften, Heft 478, 1997.
- [2] M. von Scheven, E. Ramm, and M. Bischoff, “Quantification of the redundancy distribution in truss and beam structures”, *International Journal of Solids Structures*, vol. 213, pp. 41–49, 2021, doi: [10.1016/j.ijsolstr.2020.11.002](https://doi.org/10.1016/j.ijsolstr.2020.11.002).
- [3] D. Forster and M. von Scheven, “The redundancy matrix as a performance indicator for structural assessment”, *Journal of Theoretical, Computational and Applied Mechanics*, 2025, doi: [10.46298/jtcam.13574](https://doi.org/10.46298/jtcam.13574).
- [4] U. Starossek and M. Haberland, “Approaches to measures of structural robustness”, *Structure and Infrastructure Engineering*, 7:7-8, 625-631, 2010, doi: [10.1080/15732479.2010.501562](https://doi.org/10.1080/15732479.2010.501562).
- [5] X. Kou, L. Li, Y. Zhou and J. Song, “Redundancy component matrix and structural robustness”, *International Journal of Civil and Environmental Engineering*, vol. 11, no. 8, pp. 1150–1155, 2017.
- [6] F. Geiger, *Strukturmechanische Charakterisierung von Stabtragwerken für den Entwurf adaptiver Tragwerke*, dissertation, Institut für Baustatik und Baudynamik, Universität Stuttgart, Germany, 2022, doi: [10.18419/opus-12299](https://doi.org/10.18419/opus-12299).
- [7] S. Pellegrino, “Structural computations with the singular value decomposition of the equilibrium matrix,” *International Journal of Solids and Structures*, vol. 30, no. 21, pp. 3025–3035, 1993, doi: [10.1016/0020-7683\(93\)90210-X](https://doi.org/10.1016/0020-7683(93)90210-X).