Proceedings of IASS-IWSS 2026 – September 14-18, 2026, Turin, Italy
"R-Evolution of Shapes: Sustainability, Re-Use, and New Design Paradigms"
Annual Symposium of the International Association for Shell and Spatial Structures
3rd Italian Workshop on Shell and Spatial Structures
A. Manuello Bertetto, S. Gabriele, F. Marmo, A. Micheletti (eds.)

On Graphical Methods Applied to Domes with an Oculus

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Abstract

The dome containing a central oculus is an ancient and widespread structural typology. This paper considers the load-resisting behaviour of such structures subject to different loading conditions, and especially asymmetric loading. The structural typology is investigated through the use of methods from graphic statics and rigidity theory. Closed, convex and smooth surfaces such as complete hemispherical domes are known to be rigid via Cohn-Vossen and rigidity theory and are thus able to resist any loading condition via membrane action alone. This is not the case once large openings such as an oculus are introduced, and such structures are mechanisms unless bending may also be resisted. Membrane action alone is thus insufficient. This paper makes use of graphic statics and discretised surfaces of the structure to derive linear-piecewise Airy stress functions via the Force Density Method (FDM). These are then used to solve Pucher's equation. It is demonstrated that classical results for the hemispherical dome continuum may be reproduced via the proposed methodology, for the discretised structure. The Airy stress function for an antisymmetrically loaded dome is subsequently derived and analysed to determine bending at the oculus edge, which is verified via finite element models. Despite their prevalence, domes with oculi still present interesting characteristics and behaviour from a structural perspective.

Keywords: graphic statics, form-finding, Maxwell, gridshell design, dome structures, historical structures, Force Density Method, structural design

1. Introduction

The dome, defined here as a structural, convex shell of revolution able to resist loads, is a historically ubiquitous structural typology used to form roof structures often for large spans (of 20m and beyond). There is evidence of its use spanning back millennia to some of the earliest recorded construction. Historically, unreinforced masonry (brick or stone) construction was commonly used to form domes relatively easily, with or without centring. These materials can only accommodate small tension forces and are therefore likely to crack where they occur, but this is not necessarily the case for modern shells using materials such as steel and reinforced concrete.

As in other shell structures, domes rely on membrane action to resist loading, while bending is limited to allow for reduced structural depth. This paper considers the behaviour of dome structures with the additional feature of a central oculus. This element, clearly a common feature in historical domes such as the Pantheon and various other examples, has significant effects on the typology's ability to carry loads and its response to different load cases.

It is also known via rigidity theory and Cohn-Vossen [9] that closed, convex and smooth surfaces such as complete domes, are rigid. This would not apply to a dome with a large opening such as an oculus. CJK Williams, in [8], identifies synclastic shells that contain large openings (specifically mentioning the example of a dome with an oculus), as being susceptible to inextensional deformation. If this is the

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case, then they are mechanisms and bending action is also required for equilibrium under certain loading conditions. This paper seeks to investigate these structures, making use of methods from graphic statics and rigidity theory. Can such methods provide further insight into the structural behaviour of domes with oculi?

2. Graphic statics

Graphic statics refers to 19th-century graphical methods for deriving internal forces in loaded two-dimensional trussed structures through graphical constructions, which developed in response to the emergence of early truss structures [1]. The theory underpinning graphic statics was developed by Culmann, Maxwell, Rankine and others, based on foundational work in projective geometry by Poncelet [14]. It is founded on the reciprocal nature of form and force. Graphical constructions of the internal forces in a 2D truss may be derived from its form, or similarly, a structural form may be constructed graphically from a corresponding set of internal forces in a trussed structure.

Maxwell further observed that 2D trusses are the two-dimensional projection of 3D plane-faced polyhedra, which are continuous piecewise-linear Airy stress functions [15]. The axial force in the projected element can be determined from the change in slope between adjacent faces at the corresponding shared boundary of the Airy stress function. Planar regions between boundaries are regions of no axial force.

Continuous, curving, Airy stress functions, rather than the piecewise-linear ones described above, may describe the stress states in 2D continuous elastic bodies, rather than in trussed frameworks [7]. They are often referred to in shell theory as a means of satisfying horizontal equilibrium of a given loaded shell [8, 16]. Full equilibrium under vertical loading is achieved by satisfying Pucher's equation below. This paper, makes use of graphic statics and therefore of the discretised, piecewise-linear case: continuous surfaces are discretised into triangulated gridshells. The discretisation may impact results and must be treated with caution. Nevertheless, results for the discrete case may be seen to correspond with those of the continuum.

3. Pucher's equation

The relationship between the Airy stress function (ϕ), shell geometry (F) and vertical loading (q) is given by the second-order partial differential equation below, attributed to Pucher by Timoshenko [3].

$$-q = \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 F}{\partial x^2} \tag{1}$$

If solved, the equation demonstrates that equilibrium is satisfied via membrane action alone, and the resulting membrane stress field is statically determinate. In practice, compatibility requirements may impose a degree of bending in the shell. (Here, however, we are dealing with equilibrium solutions only and compatibility is not necessarily satisfied.) The existence of a solution to Pucher's equation depends on shell form and boundary conditions, and is not guaranteed. In the absence of a solution, some degree of bending may be required to satisfy equilibrium.

Discretised surfaces representing plane-faced, piece-wise linear Airy stress functions may be coupled with the Force Density Method (FDM) [6] to solve Pucher's equation 1. The method offers several advantages analytically: surface continuity requirements are removed and equilibrium is resolved at specific nodes at discrete locations rather than for a continuous field. By doing so, a sparse linear system is used which can be solved algebraically, directly and without iteration.

The limitations of the proposed method include surface discretisation affecting results, outputs limited to approximations of surfaces rather than "complete" results (since equilibrium is only solved at nodes, as above). Refer to Miki [19] for an alternative method for dealing with the continuous case.

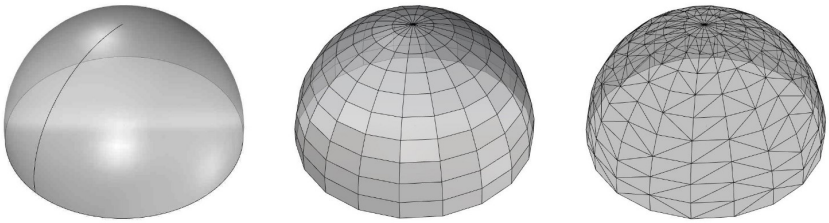


Figure 1: Surface discretisation of hemispherical surface: original surface (left); discretisation along principal curvatures resulting in planar quads generally (centre); discretisation into triangulated mesh (right).

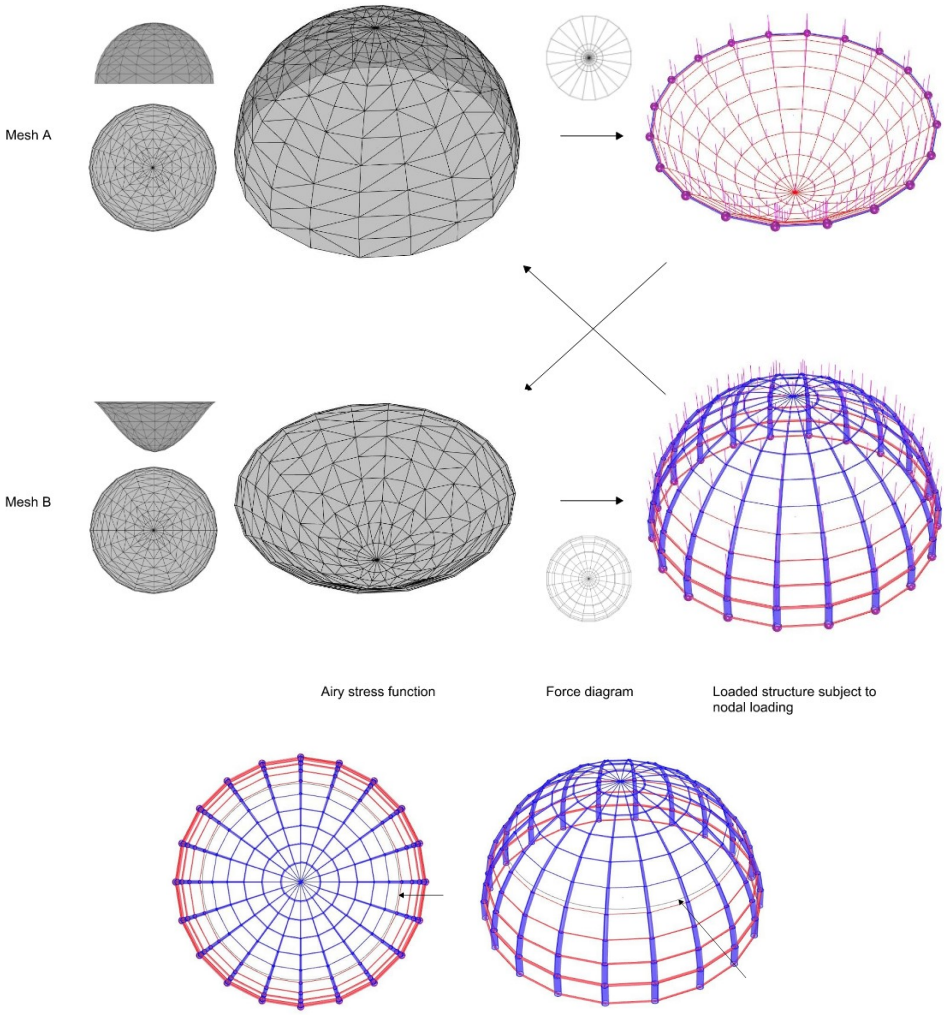


Figure 2: Complete hemispherical structure (A), associated Airy stress function (B) and force diagrams: B is obtained by treating A as Airy stress function and solving for forces for given nodal loads using FDM (first row); forces in A for the same loading are solved by treating B as Airy stress function. Nodal loads are based on tributary areas of the mesh (i.e. uniform loading - similar to those computed for the self-weight of dome of constant thickness) (second row). Hoop forces in complete hemispherical structure subject to uniform loading seen to respect theoretical transition of approximately 52°indicated in grey and by arrows (bottom).

4. Structural action of hemispherical domes subject to vertical loading

The numerical analysis of the membrane stress states in (continuous) hemispherical domes subject to vertical loading, has been developed by figures such as Navier [10], Rayleigh [12], Clapeyron and Lamé [11], Love [2] and others [1]. Timoshenko [3] presents the standard results familiar to many engineers.

Prior to this, the relevant structural theory was geometrical in nature and closely related to that of the masonry arch, and Hooke's observed relationship between the inverted catenary and the arch [5]. While determining the geometry of lines of thrust through the use of inverted catenary forms is a valid method for determining equilibrium in compression shell structures where tension is inadmissible (e.g. unreinforced masonry), this is not necessarily the case for shells where tension is admissible such as in steel or reinforced concrete. In such structures, equilibrium may be maintained through the formation of tension hoops or alternative three-dimensional load paths that incorporate tension.

4.1. Hemispherical domes subject to uniform vertical load

Timoshenko [3] provides results for the continuous and complete hemispherical dome subject to self-weight. Stresses are given in the meridional (2) and hoop (3) directions [3]. The well-known result implied by (3) is that hoop stresses are negative (compression) for meridional angles below circa 51.82°, at which they are zero, and beyond which they become positive (tension).

$$N_\phi = -\frac{Rq}{1+\cos\phi} \tag{2}$$

$$N_\theta = Rq(\frac{1}{1+\cos\phi} - \cos\phi) \tag{3}$$

4.2. Hemispherical domes subject to asymmetric vertical load

The standard analytical method for dealing with asymmetric loading of a hemispherical dome (or other shell of revolution) is to conduct a Fourier decomposition of the loading. This is described by Calladine [16] and Flügge [13], and is laborious; finite element analysis is typically preferred. Timoshenko [3] includes a single example which uses the first harmonic to solve asymmetric loading due to wind. (As noted, from rigidity theory, we know that for the complete hemisphere, membrane solutions are in fact theoretically possible for any form of loading.)

4.3. Hemispherical domes with an oculus

Timoshenko [3] and Flügge [13] consider hemispherical domes with a central opening subject to uniform vertical loading as well as a concentrated (uniform) ring load at the opening. A membrane solution for this case is therefore identified. Timoshenko does not consider asymmetric loading while Flügge and others discuss the analytical treatment of edges and associated difficulties. In the asymmetric case, Fourier decomposition of the loading is again required, but the presence of the oculus introduces an additional boundary that substantially increases the complexity of the membrane formulation.

5. Proposed methodology

Note that in Pucher's equation, form (F) and Airy stress function (ϕ) are interchangeable [20] [17]. This allows the form to be treated as Airy stress function or vice versa, and for an identical process used to solve the PDE given a known geometry and loading. In this case, FDM is used to solve for F or ϕ once force densities are computed from the input surface. This interchangeability of form and Airy stress function is leveraged in the examples presented here. The proposed method is outlined below and is similar to that described in [20].

Firstly, the shell surface is discretised using a triangular tessellation, to create a mesh (geometry A) where mesh edges represent structural elements. These elements may be understood as those of a gridshell or

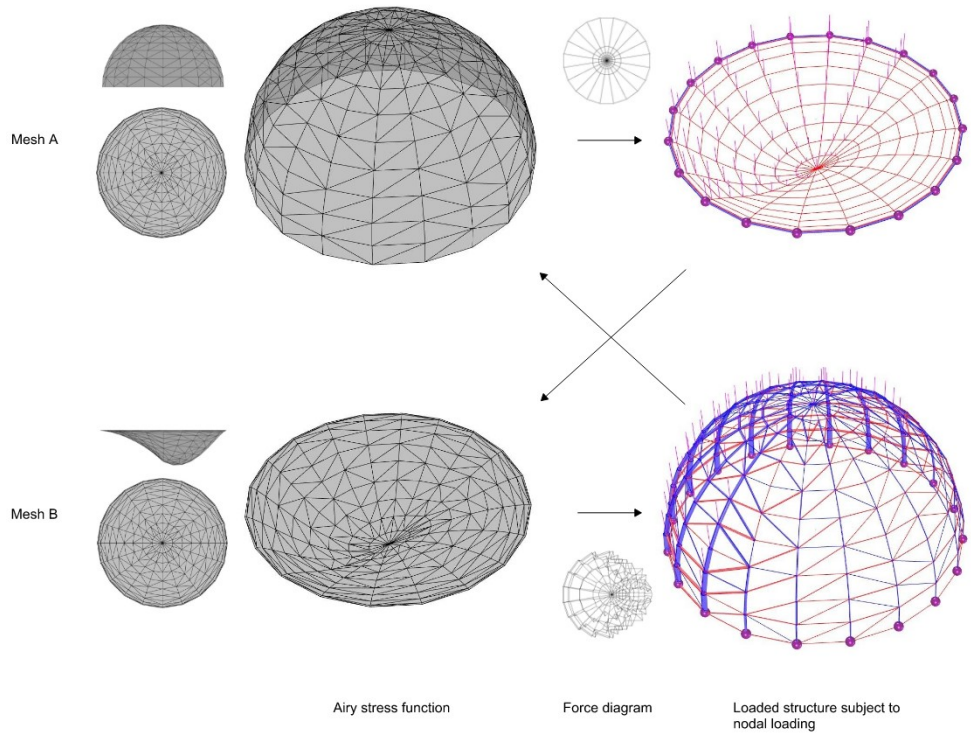


Figure 3: Complete hemispherical structure (A), associated Airy stress function (B) and force diagrams for asymmetric loading (- half surface loaded). Diagonal members in A observed to resist forces.

the positions at which we are measuring forces within a continuous shell. This new geometry A is form F in equation 1, but is still treated as an Airy stress function by making use of the interchangeability described above. Mesh A can then be used to construct a 2D force diagram, following the construction described in [21] and others, where the forces describe horizontal equilibrium in some other 3D geometry, geometry B. When considering geometry A as an Airy stress function, since it is discretised, the regions of the mesh (the mesh faces) between edges are planar and contain no change in slope and therefore no force. Unlike in the continuum (like the original shell surface prior to discretisation), therefore, forces are concentrated at specific elements represented by mesh edges. For a given vertical loading condition (q), and for element force densities calculated using the forces from the force diagram and horizontally projected lengths of elements, the new geometry B may be computed using FDM. For this process, nodes at the boundary are identified as fixed while the other nodes' z-coordinate is determined using FDM to satisfy equilibrium. Note that geometries A and B have the same horizontal projections. They may be understood as being interchangeable in equation 1, but are not reciprocal in the projective geometry sense as described by Poncelet [14].

So far, geometry B and associated element forces for a given loading have been computed. The final step in the proposed method is to determine the forces in initial geometry A. From the interchangeability of form (F) and Airy stress function (ϕ), geometry B in interpreted as Airy stress function and used to construct a second force diagram, which describes horizontal equilibrium in geometry A. The forces in A are calculated using the force densities of the projection on the actual element lengths of A. Furthermore, using the force densities calculated from the second force diagram in FDM, geometry A itself can be computed as an output thus verifying results and that a correct solution pair of geometries have been found for solving Pucher's equation (1).

6. Results

6.1. Complete Hemisphere

The complete hemispherical dome is considered first, subject to two separate load cases: a symmetrical uniform load case similar to self-weight loading (assuming minimal variation of weight across the structure's surface); an asymmetric load case where the surface is split into halves and one half assigned uniform vertical downward loading. Results are presented in fig. 2 and fig. 3 and include form F, Airy stress function ϕ and associated force diagrams.

Note for the uniform load case, results closely match those expected for the continuum described in above, with hoop stresses transitioning from compressive to tensile forces at approximately 52°. This result is dependent on the discretisation used which in this case follows principal stress directions in the hoop and meridional directions. Note that under uniform vertical loading, triangulating elements not aligned with principal directions do not carry any load, and the active elements lie on the boundaries of the planar quads forming the hemisphere.

Subject to asymmetric loading, it can be seen that axial forces appear in all elements including hoop, meridional and diagonal elements (fig. 3). This occurs since principal stress directions are no longer aligned with the discretisation. An alternative discretisation could be selected aligned with principal stresses and where elements would also lie on the boundaries of planar quads without triangulation being required. This is load case-specific, however, and would not be funicular for other load cases without triangulation. Even a small asymmetry in loading results in significant engagement of the diagonals.

The Airy stress functions for the hemispherical shell for either case may be considered as structures in their own right (due to the interchangeability discussed above). Both of these geometries treated as structures have identical hemispherical Airy stress functions, but are subject to different loading. Since the Airy stress function is synclastic, the structures can be expected to be entirely in tension, or entirely in compression, except at the boundary.

6.2. Hemispherical Dome with Oculus

Two similar load cases are again considered for the hemispherical dome with a centrally-positioned oculus. The oculus dimension is not varied in this study, but this parameter is not expected to have significant impact on results presented here. A triangulated, discretised geometry is again used, with the oculus also meshed using elements that lie on a plane, so that when this geometry is treated as Airy stress function, the corresponding elements contain no force. The edges for these are theoretical elements used to deal with the opening, and care is required when considering them in the FDM process and its outputs, since they form part of an independent lift and can be used to prestress the oculus edge. Results are presented in fig. 4 and fig. 5. The load case used in this instance is antisymmetric, which through superposition can be combined with uniform loading to create the half-loaded surface of 3. (This is the subject of future work.)

As can be seen in fig. 4, for uniform vertical loading, a membrane solution is found that does not engage the theoretical elements across the oculus. This is because in the associated Airy stress function they lie on the same plane as does the oculus edge, and the theoretical elements may thus be ignored; no bending of the oculus is required to resolve horizontal equilibrium. Furthermore, as in the complete hemispherical dome, beyond the oculus only hoop and meridional elements are engaged while triangulating elements are not required to resist load. The existence of a membrane solution for this load case in the continuum case is noted by Timoshenko and Flügge.

As is evident in fig. 6 for the antisymmetric case, the elements in the Airy stress function corresponding to the oculus do not lie in the same plane: the theoretical elements across the oculus must be engaged or a discontinuity introduced in the Airy stress function [21]. Without these elements across the oculus, bending is required in oculus edge elements to resolve horizontal equilibrium of the projected shell.

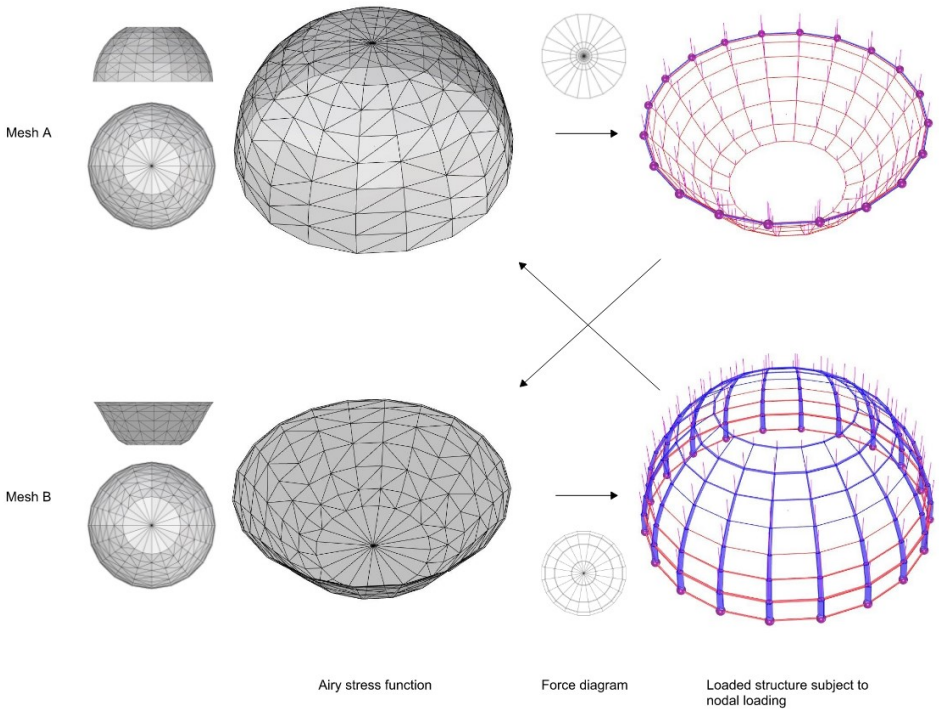


Figure 4: Dome surface with oculus (A), associated Airy stress function (B) and force diagrams for uniform loading. Note the oculus is also meshed for the purpose of the analysis.

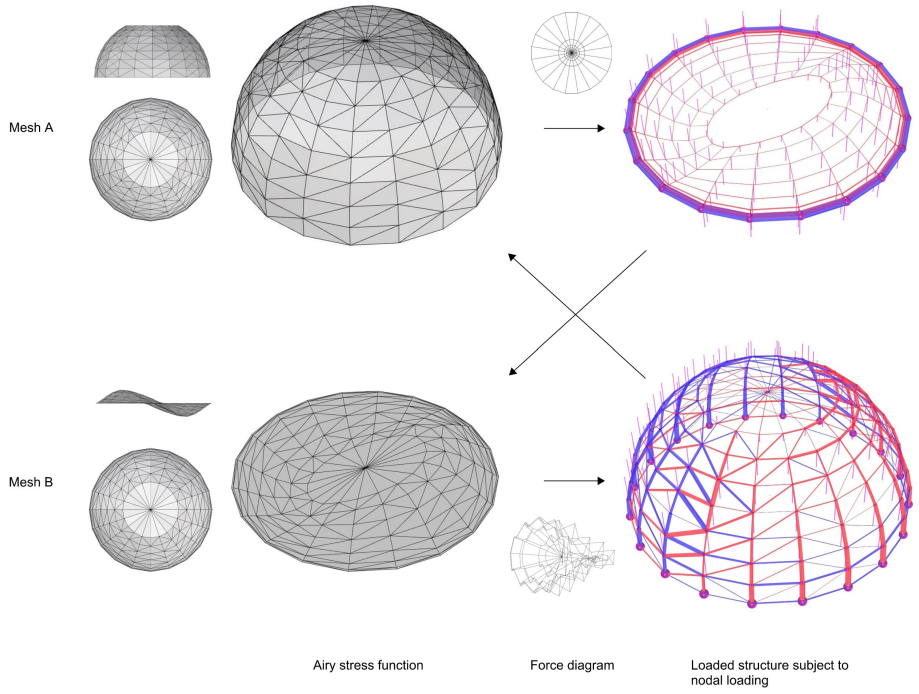


Figure 5: Dome surface with oculus (A), associated Airy stress function (B) and force diagrams for antisymmetric uniform loading.

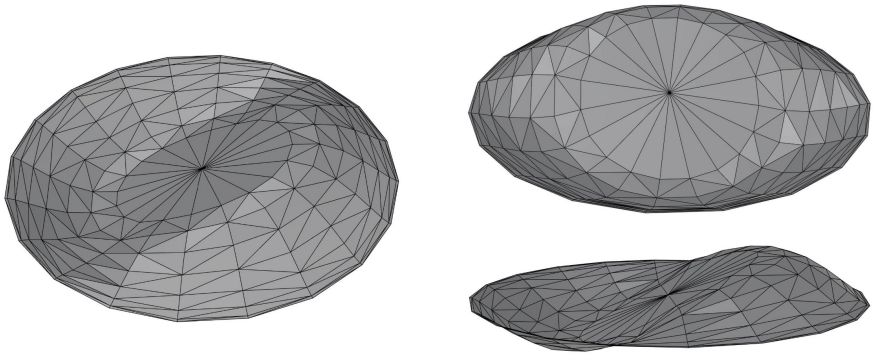


Figure 6: Airy stress function for dome with oculus subject to antisymmetric uniform loading.

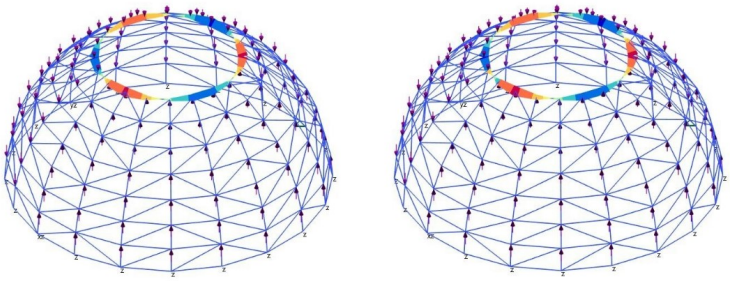


Figure 7: Oasys GSA analysis of hemispherical dome with oculus subject to antisymmetric load showing major (left) and minor (right) axis bending in oculus element.

7. Results verification

A finite element model of identical geometry and loading has been used to compare the above results to. Axial force magnitude and distribution are observed to be practically identical. Bending in the oculus edge for the antisymmetric case may also also be investigated further and compared. In three dimensions, this bending resolves to both lateral and vertical bending of the oculus ring element. The ring element must therefore be able to resist both. This can be seen in fig. 7 of the finite element model. A vertical bending element such as a ”lip” at the oculus edge capable of resisting vertical bending is thus insufficient. It is notable that the distribution of bending at the oculus edge in the first instance, does not appear to have any geometric relationship with the applied loading. In fact, the bending corresponds to a specific mechanism of the structure which is being enabled by the antisymmetric loading. A closer look at the geometry of the Airy stress function at the region of the oculus reveals a discontinuity of similar geometry (fig. 8), corresponding with a bending distribution similar to that observed in the finite element model.

8. Rigidity theory and relevant mechanisms

As noted in the introduction, it is known via rigidity theory that closed, convex and smooth surfaces such as complete domes, are rigid [9]. Convex polyhedral surfaces obtained through triangular discretisation (as above) are likewise rigid [22]. Rigidity can be demonstrated by performing singular value decomposition (SVD) on the equilibrium matrix for the discretised framework [23]. This method also allows for mechanisms to be plotted. The mechanism at the oculus edge (if bending is not permitted) is shown in fig. 9.

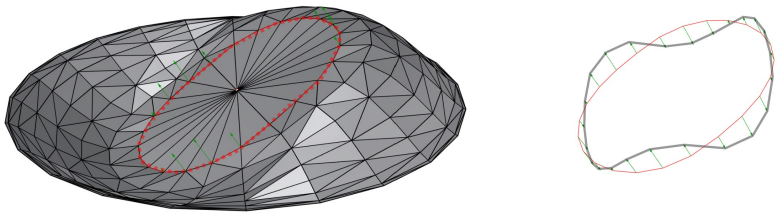


Figure 8: Non-planar Airy stress function geometry at the oculus and corresponding bending moment distribution.

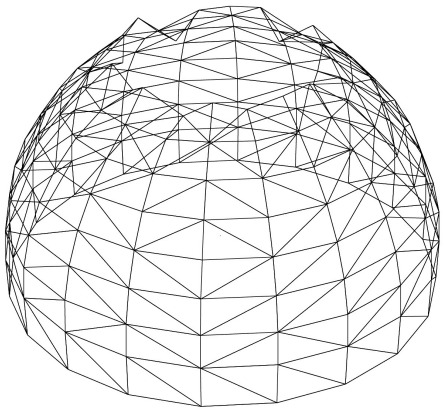


Figure 9: Oculus mechanism derived using SVD on triangular discretisation of dome surface.

9. Conclusions and further work

This paper has investigated the structural behaviour of hemispherical domes with an oculus using a combination of discretised Airy stress function, graphic statics and rigidity theory. For the complete hemispherical case subject to uniform vertical loading, results align closely with known classical solutions, confirming the validity of the approach. Airy stress functions for asymmetric vertical loading were derived for both the complete hemispherical dome and that with an oculus. It was shown that for the given loading, bending of the oculus is required and that this can be determined by inspecting the Airy stress function. The distribution of bending at the edge was shown to match that obtained via finite element analysis. Finally, the mechanisms at the oculus were derived using SVD applied to the equilibrium matrix for the discretised framework. Further work will make more detailed consideration of the membrane action load-resisting behaviour of this structural typology, and will expand on the observed bending distribution and its graphical analysis, which is not exclusive to the antisymmetric loading condition.

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